MAT 219 (lecture 9) | §7.6 Complex Eigenvalues → 2×2 & 3×3 examples → Drawing Phase Portraits | **Next:** §7.7 Fundamental Matrix. → Jordan Form of Matrix; §7.8 Repeated Eigenvalues. | 1

§7.6 Complex Eigenvalues

Previously in MAT 219:

$\boxed{\underline{x}' = A\underline{x}}$ with $\mathbb{C}$-conjugate eigenvalues $\boxed{\lambda = a \pm bi}$

$\Rightarrow$ eigenvectors are also $\mathbb{C}$-conjugate $\boxed{\underline{v} = \underline{\alpha} \pm \underline{\beta} i}$

Last time: Look at Real and Imaginary parts.

$$e^{(a \pm bi)t}(\underline{\alpha} \pm \underline{\beta} i) = e^{at} \cdot e^{\pm bti}(\underline{\alpha} \pm \underline{\beta} i)$$

$$= e^{at}(\cos bt \pm i \sin bt)(\underline{\alpha} \pm \underline{\beta} i)$$

$$= \boxed{e^{at}(\underline{\alpha} \cos bt - \underline{\beta} \sin bt)}$$

$$\pm \boxed{e^{at}(\underline{\alpha} \sin bt + \underline{\beta} \cos bt)} i$$

→ The real and imaginary parts are each REAL solutions by themselves (without i)

Solution:

$$\boxed{\underline{x} = c_1 e^{at}(\underline{\alpha} \cos bt - \underline{\beta} \sin bt) + c_2 e^{at}(\underline{\alpha} \sin bt + \underline{\beta} \cos bt)}$$

EX: The system $\underline{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \underline{x}$

has eigenvalues $\lambda = 1 \pm 2i$ (conjugate)

with eigenvectors $\underline{v} = \begin{bmatrix} 1 \\ \mp i \end{bmatrix}$ (conjugate)

Write $\lambda = \underbrace{\boxed{1}}_{a} + \underbrace{\boxed{2}}_{b} i$ & $\underline{v} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{\alpha}} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\underline{\beta}} i$

Solution:

$$\underline{x} = c_1 e^{t}\left(\overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 2t}^{\underline{\alpha}\cos bt} - \overbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin 2t}^{\underline{\beta}\sin bt} \right)$$

$$+ c_2 e^{t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos 2t \right)$$

$$\boxed{x = c_1 e^{t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix}}$$

Note: Only need to use one of the eigenvalue/eigenvector pairs (because they are conjugate)

Initial Value Problem:

EX: $\underline{x}' = \begin{bmatrix} 7 & -4 \\ 5 & -1 \end{bmatrix} \underline{x}$ with $\underline{x}(0) = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

Eigenvalues

$$\lambda^2 - 6\lambda + 13 = 0$$
$$(\lambda - 3)^2 + 4 = 0$$
$$\boxed{\lambda = 3 \pm 2i}$$

Solution has

$$e^{(3+2i)t} = e^{3t} \cdot e^{2ti} = e^{3t}(\cos 2t + i \sin 2t)$$

Eigenvectors (only need to compute one)

$\lambda = 3+2i \Rightarrow \begin{bmatrix} 7-(3+2i) & -4 \\ 5 & -1-(3+2i) \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} 4-2i & -4 \\ 5 & -4-2i \end{bmatrix} \underline{v} = \underline{0} \qquad \underline{v} = \begin{bmatrix} 4 \\ 4-2i \end{bmatrix}$$

$\times \frac{1}{2}$

$$\boxed{\underline{v} = \begin{bmatrix} 2 \\ 2-i \end{bmatrix}} \Rightarrow \boxed{\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i}$$

General Soln:

$$\underline{x} = c_1 e^{3t}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin 2t\right) + c_2 e^{3t}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos 2t\right)$$

$$\boxed{\underline{x} = c_1 e^{3t} \begin{bmatrix} 2\cos 2t \\ 2\cos 2t + \sin 2t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2\sin 2t \\ 2\sin 2t - \cos 2t \end{bmatrix}}$$

(EX continued)

Initial Values

$$\begin{bmatrix} -4 \\ -7 \end{bmatrix} = \underline{x}(0) = c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 2+0 \end{bmatrix} + c_2 \cdot 1 \cdot \begin{bmatrix} 0 \\ 0-1 \end{bmatrix}$$

$$\begin{cases} -4 = 2c_1 \longrightarrow \boxed{c_1 = -2} \\ -7 = 2c_1 - c_2 \longrightarrow -7 = -4 - c_2 \end{cases}$$

$$\boxed{c_2 = 3}$$

Answer:

$$\underline{x} = -2e^{3t} \begin{bmatrix} 2\cos 2t \\ 2\cos 2t + \sin 2t \end{bmatrix} + 3e^{3t} \begin{bmatrix} 2\sin 2t \\ 2\sin 2t - \cos 2t \end{bmatrix}$$

$$\boxed{\underline{x} = e^{3t} \begin{bmatrix} 6\sin 2t - 4\cos 2t \\ 4\sin 2t - 7\cos 2t \end{bmatrix}}$$

WARNING: In my computations, I always use the $a+bi$ eigenvalue for finding eigenvector.

$\Rightarrow$ If you use $a-bi$ then solution has

$$\cos(-bt) = \cos(bt)$$
$$\sin(-bt) = -\sin(bt)$$

Note: Large systems can have $\mathbb{R}$ & $\mathbb{C}$ eigenvals.

$\Rightarrow$ General solution is combination of all "little" solutions.

EX: Suppose $\underline{x}' = A\underline{x}$ is 5×5 with

- $\lambda = 2 \rightsquigarrow \underline{v} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \longrightarrow e^{2t}\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$

- $\lambda = 5+i \rightsquigarrow \underline{v} = \begin{bmatrix}1\\-i\\2+i\\1-3i\\4\end{bmatrix} \longrightarrow e^{5t}\left(\begin{bmatrix}1\\0\\2\\1\\4\end{bmatrix}\cos t - \begin{bmatrix}0\\-1\\1\\-3\\0\end{bmatrix}\sin t\right)$

  and

- conjugate: $e^{5t}\left(\begin{bmatrix}1\\0\\2\\1\\4\end{bmatrix}\sin t + \begin{bmatrix}0\\-1\\1\\-3\\0\end{bmatrix}\cos t\right)$

- $\lambda = -3+2i \rightsquigarrow \underline{v} = \begin{bmatrix}1\\0\\2-i\\4i\\3\end{bmatrix} \longrightarrow e^{-3t}\left(\begin{bmatrix}1\\0\\2\\0\\3\end{bmatrix}\cos 2t - \begin{bmatrix}0\\0\\-1\\4\\0\end{bmatrix}\sin 2t\right)$

  and

- conjugate: $e^{-3t}\left(\begin{bmatrix}1\\0\\2\\0\\3\end{bmatrix}\sin 2t + \begin{bmatrix}0\\0\\-1\\4\\0\end{bmatrix}\cos 2t\right)$

General solution is sum of these (times c_1, c_2, etc)

EX: $\underline{x}' = \begin{bmatrix}-1 & 0 & -5\\ 2 & 3 & 1\\ 2 & 0 & 5\end{bmatrix}\underline{x}$

Eigenvalues

$$\det\begin{bmatrix}-1-\lambda & 0 & -5\\ 2 & 3-\lambda & 1\\ 2 & 0 & 5-\lambda\end{bmatrix} = 0$$

$$0 = (-1-\lambda)(3-\lambda)(5-\lambda) + \cancel{(0)(1)(2)} + \cancel{(-5)(2)(0)}$$
$$- \left(\cancel{(-1-\lambda)(1)(0)} + \cancel{(0)(2)(5-\lambda)} + (-5)(3-\lambda)(2)\right)$$

$$0 = (3-\lambda)\left((-1-\lambda)(5-\lambda) - (-5)(2)\right)$$

$$0 = (3-\lambda)(\lambda^2 - 4\lambda + 5) \qquad ((\lambda-2)^2 + 1)$$

$$\boxed{\lambda = 3,\ 2 \pm i}$$

Eigenvectors $\quad \lambda = 3$

$$\begin{bmatrix}-1-3 & 0 & -5\\ 2 & 3-3 & 1\\ 2 & 0 & 5-3\end{bmatrix}\underline{v} = 0 \rightsquigarrow \begin{cases}-4x - 5z = 0\\ 2x + z = 0\\ 2x + 2z = 0\end{cases}$$

$\longrightarrow$ let ~~$x = 1$~~ then

~~$-4 - 5z = 0 \rightsquigarrow z = -4/5$~~
~~$2 + z = 0 \rightsquigarrow z = -2$~~
~~$2 + 2z = 0 \rightsquigarrow z = -1$~~

Not possible!

$\longrightarrow$ let $y = 1$ then

$$-4x - 5z = 0$$
$$2x + z = 0$$
$$-\ 2x + 2z = 0$$
$$-z = 0 \rightsquigarrow x = 0.$$

$$\boxed{\underline{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}}$$

(Ex continues) $\lambda = 2+i$

$$\begin{bmatrix} -1-(2+i) & 0 & -5 \\ 2 & 3-(2+i) & 1 \\ 2 & 0 & 5-(2+i) \end{bmatrix} \underline{v} = \underline{0}$$

$$\Downarrow$$

$$\begin{bmatrix} \boxed{(-3-i)x \qquad -5z = 0} \\ \boxed{2x + (1-i)y + z = 0} \\ 2x \quad + (3-i)z = 0 \end{bmatrix}$$

If we let $x=5$ then z will be $3+i$

let $\underline{x=5}$ then

$(-3-i)5 - 5z = 0$

$\underline{z = 3+i}$

$2 \cdot 5 + (1-i)y + (3+i) = 0$

$y = \frac{(-13-i)}{(1-i)} \frac{(1+i)}{(1+i)} = \frac{(-13+1)+(-13-1)i}{2} = \underline{-6-7i}$

So $\underline{v} = \begin{bmatrix} 5 \\ -6-7i \\ 3+i \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -7 \\ 1 \end{bmatrix} i}$

Answer:

$$\boxed{\begin{aligned} \underline{x} = c_1 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &+ c_2 e^{2t} \left(\begin{bmatrix} 5 \\ -6 \\ 3 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -7 \\ 1 \end{bmatrix} \sin t \right) \\ &+ c_3 e^{2t} \left(\begin{bmatrix} 5 \\ -6 \\ 3 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ -7 \\ 1 \end{bmatrix} \cos t \right) \end{aligned}}$$

(That was not very pleasant.)

What about phase portraits?

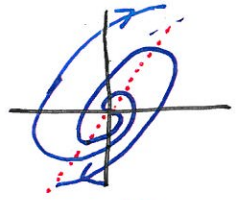
Spirals.

$$\left.\begin{aligned} \lambda &= a \pm bi \\ \underline{v} &= \underline{\alpha} \pm \underline{\beta} i \end{aligned}\right\} \Longrightarrow \begin{aligned} \underline{x} = c_1 &\boxed{e^{at}} (\underline{\alpha} \cos bt - \underline{\beta} \sin bt) \\ + c_2 &\boxed{e^{at}} (\underline{\alpha} \sin bt + \underline{\beta} \cos bt) \end{aligned}$$

e^{at} decides if solutions grow or shrink

- Real(λ) $\begin{cases} a > 0 \Longrightarrow \text{Spiral out to } \infty \\ a = 0 \Longrightarrow \text{Ellipse} \\ a < 0 \Longrightarrow \text{Spiral in to } 0 \end{cases}$
- Im(λ) $\quad b \Longrightarrow$ Spiral quickly/slowly
- Real($\underline{v}$) $\quad \underline{\alpha} \Longrightarrow$ Stretches the spiral
- Im($\underline{v}$) $\quad \underline{\beta} \Longrightarrow$ Spiral clockwise (+) or counter-clockwise (−)

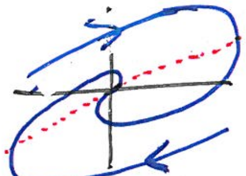
EX: $\underline{x}' = \begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix} \underline{x}$



Eigenvalues $\lambda = 1 \pm i$ — spiral out.

Eigenvector $\lambda = 1+i \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$ — stretch along $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

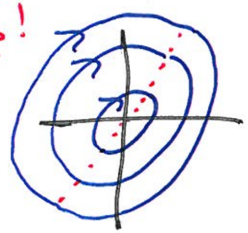
EX: $\underline{x}' = \begin{bmatrix} -5 & 5 \\ -1 & -1 \end{bmatrix} \underline{x}$



Eigenvalues $\lambda = -3 \pm i$ — spiral in

Eigenvector $\lambda = -3+i \rightsquigarrow \underline{v} = \begin{bmatrix} 5 \\ 2+i \end{bmatrix}$ — stretch along $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

EX $\underline{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix} \underline{x}$



Eigenvalues $\lambda = \pm i$ — Real part = 0 $\Rightarrow$ ellipses!

Eigenvector $\lambda = i \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$ — stretch along $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\rightarrow$ These were all clock-wise rotation

Sometimes solutions do reverse

Rotation direction comes from sign of imaginary part of $\underline{v}$ when you use $\lambda = a \underline{\underline{+}} bi$